**Quasi-Polynomial Mapping based Rootfinder**

QPmR v.2
Created by Tomas Vyhlidal, CTU in Prague
http://www.cak.fs.cvut.cz/algorithms/qpmr

\[ \text{QPmR(Region, P, D)} \]

Finds all zeros of the quasi-polynomial

\[ \text{QP}(s) = (P(1,1)*s^n+...+P(1,n)*s+P(1,n+1))\exp(-D(1)*s) + \\
(P(2,1)*s^n+...+P(2,n)*s+P(2,n+1))\exp(-D(2)*s) + \\
... + (P(N,1)*s^n+...+P(N,n)*s+P(N,n+1))\exp(-D(N))*s) \]

located in the complex plane region defined by

\[ \text{Region} = [\text{real_min real_max imag_min imag_max}] \]

The other two function inputs are:

- \( P \) - N by n matrix of polynomial coefficients, \( n \) is the maximum power of \( s \) in the quasi-polynomial, \( N \) is the number of delays.
- \( D \) - vector of different (positive) delay values of size \( N \). One of the delays should be equal to 0.

The quasi-polynomial can also be defined in the function handle form \( \text{Fun} \).
Then, the QPmR function syntax is

\[ \text{QPmR(Region, Fun)} \]

The QPmR function output is the vector of all quasi-polynomial zeros located in

the given region. The method is based on mapping the quasi-polynomial zero level curves over the complex plane region with the adaptation of the mapping grid.

NaN as the function output indicates failure of the adaptation algorithm.
In this case, the region Reg should be reduced. Alternatively, the grid can be assigned manually using the extended function modes

\[ \text{QPmR(Region, P, D, e, ds, gr)} \] or \[ \text{QPmR(Region, Fun, e, ds, gr)} \]

where the additional input parameters are

- \( e \) - computation accuracy. If \( e=-1 \), then \( e=1e-6*ds \) (the same accuracy is considered if \( e \) parameter is not given, as it is above).
- \( ds \) - grid step for mapping the zero-level curves. If \( ds=-1 \), the grid step is adjusted automatically.
- \( gr \) - graphical representation of the results. If \( gr=1 \), results are visualized in plots (default \( gr=0 \)). If the quasi-polynomial is given in \( P \) and \( D \), the spectrum distribution diagram and the asymptotic exponentials of the spectra are visualized. If the
quasi-polynomial is neutral, also the spectrum of the associated difference equation is computed and its safe upper bound is determined and visualized. If gr=1, the information on the grid adaptation is provided in the command window.

\[ [R,Y] = \text{(Region, P, D, \ldots)} \] provides the following outputs

- \( R \) - computed zeros of the quasi-polynomial (NaN indicates the algorithm failure)
- \( Y \) - structure with summary of the QPmR results, particularly
  - \( Y.\text{zeros} \) - computed zeros of the quasi-polynomial (available also if \( R=\text{NaN} \))
- \( Y.\text{flag} \) - result correctness flag.
  - \( Y.\text{flag}=1 \) - the positive result of cross-checking implies that the zeros are computed correctly.
  - \( Y.\text{flag}=0 \) - method failure: either the region is too large or there are multiple or dense roots. Next, the function can also be ill-conditioned. Try to reduce the region.
  - \( Y.\text{flag}=-1 \) - method failure: too large grid, which causes Newton's iterations failure. The grid size ds should be reduced (manually, if needed).
- \( Y.\text{accuracy} \) - accuracy estimate of the computed zeros
- \( Y.\text{asympt} \) - parameters \([\text{mi}, \text{c}]\) of the asymptotic exponentials of the root chains \( \text{real}=c(k)-\text{mi}(k)\log(\text{imag}) \);
- \( Y.\text{function} \) - quasi-polynomial in the function handle form
- \( Y.\text{grid} \) - final grid size

Additional outputs for neutral quasi-polynomials

- \( Y.\text{DEzeros} \) - computed zeros of the associated difference equation
- \( Y.\text{DEflag} \) - result correctness flag (the same as above)
- \( Y.\text{DEaccuracy} \) - accuracy estimate of the computed zeros
- \( Y.\text{DEupbound} \) - safe upper bound on the spectrum of difference equation
- \( Y.\text{DEfunction} \) - difference equation in the function handle form
- \( Y.\text{DEgrid} \) - final grid size

\[ [R,Y] = \text{(Region, Fun, \ldots)} \] provides the following outputs

- \( R \) - computed zeros of the quasi-polynomial (NaN indicates the algorithm failure)
- \( Y \) - summary of the QPmR function results.
- \( Y.\text{zeros} \) - computed zeros of the quasi-polynomial
- \( Y.\text{flag} \) - result correctness flag - the same as above
- \( Y.\text{accuracy} \) - accuracy estimate of the computed zeros
- \( Y.\text{grid} \) - final grid size

Remark 1: in the QPmR(Reg,Fun,\ldots), the function can be used for computing roots of general analytical functions, e.g. fractional polynomials or quasi-polynomials.

Remark 2: in the automatic adjustment of the grid size (ds=-1), first, the grid size is set to \( \text{ds}=(\text{Reg}(2)-\text{Reg}(1))\times(\text{Reg}(4)-\text{Reg}(3))/\text{Ns} \), where \( \text{Ns}=1000 \) (or \( \text{Ns}=500 \) for more complex functions). If not sufficient, the grid size is up to twice time reduced by the factor of four. If not sufficient, the region is divided first to four and then up to 16
Sub-regions if needed, and the QPmR runs recursively in two recursion levels.

Example
Find all the roots of the quasi-polynomial
\[ Q(s) = (1.5s^3+0.2s^2+20.1)+(s^3-2.1s)\exp(-s\times1.3) + 
3.2s\exp(-s\times3.5)+1.4\exp(-s\times4.3) \]
located in the region \( \text{Reg=}[\text{-10 5 0 300}] \)

a) representation of the quasi-polynomial by the coefficient matrix and vectors of delays:

\[
P = \begin{bmatrix} 1.5 & 0.2 & 0 & 20.1; & 1 & 0 & -2.1 & 0 & 0 & 3.2 & 0 & 0 & 0 & 1.4 \end{bmatrix} \\
D = [0;1.3;3.5;4.3]
\]

\( R = \text{QPmR}([\text{-10 5 0 300}],P,D) \) - provides the computed zeros in the vector \( R \). No graphical outputs, accuracy and grid size are adjusted automatically.

\[ [R \ Y] = \text{QPmR}([\text{-10 5 0 300}],P,D,-1,-1,1) \) - provides the computed zeros in the vector \( R \). Structure \( Y \) contains additional information on the spectrum and its computational aspects. With graphical outputs, accuracy and grid size are adjusted automatically.

\[ [R \ Y] = \text{QPmR}([\text{-10 5 0 300}],P,D,\text{1e-8}) \) - with given accuracy \( 1\text{e-8} \). Grid size is adjusted automatically, no graphical outputs.

\[ [R \ Y] = \text{QPmR}([\text{-10 5 0 300}],P,D,\text{1e-8},0.1) \) - with given accuracy \( 1\text{e-8} \) and the fixed grid size \( 0.1 \). No graphical outputs. No grid adaptation.

b) representation of the quasi-polynomial by the function handle

\[ \text{Fun= @(s) (1.5s.^3+0.2*s.^2+20.1)+(s.^3-2.1*s).*exp(-s*1.3)+} \\
3.2.*s.*exp(-s*3.5)+1.4.*exp(-s*4.3) \]

\( R = \text{QPmR}([\text{-10 5 0 300}],\text{Fun}) \)

\[ [R \ Y] = \text{QPmR}([\text{-10 5 0 300}],P,D,-1,-1,1) \)